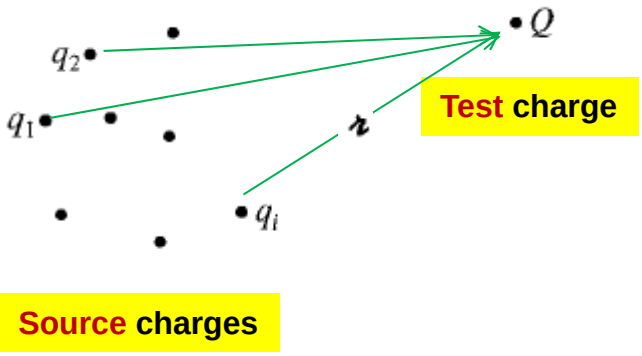
Chapter 2. Electrostatics

2 Electrostatics 58

2.1 The Electric Field . . . 58
2.1.1 Introduction . . . 58
2.1.2 Coulomb's Law . . . 59
2.1.3 The Electric Field . . . 60
2.1.4 Continuous Charge Distributions . . . 61
2.2 Divergence and Curl of Electrostatic Fields . . . 65
2.2.1 Field Lines, Flux, and Gauss's Law . . . 65
2.2.2 The Divergence of E . . . 69
2.2.3 Applications of Gauss's Law . . . 70
2.2.4 The Curl of E . . . 76
2.3 Electric Potential . . . 77
2.3.1 Introduction to Potential . . . 77
2.3.2 Comments on Potential . . . 79
2.3.3 Poisson's Equation and Laplace's Equation . . . 83
2.3.4 The Potential of a Localized Charge Distribution . . . 83
2.3.5 Summary; Electrostatic Boundary Conditions . . . 87
2.4 Work and Energy in Electrostatics . . . 90
2.4.1 The Work Done to Move a Charge . . . 90
2.4.2 The Energy of a Point Charge Distribution . . . 91
2.4.3 The Energy of a Continuous Charge Distribution . . . 93
2.4.4 Comments on Electrostatic Energy . . . 95
2.5 Conductors . . . 96
2.5.1 Basic Properties . . . 96
2.5.2 Induced Charges . . . 98
2.5.3 Surface Charge and the Force on a Conductor . . . 102
2.5.4 Capacitors . . . 103

***Introduction to Electrodynamics*, 3rd or 4rd Edition, David J. Griffiths**

2.1 The Electric Field



2.1.1 Introduction

The fundamental problem that electromagnetic theory hopes to solve is this:

→ **What force** do the **source charges (q_1, q_2, …)** exert on the **test charge (Q)?**

→ In general, **both the source charges and the test charge are in motion.**

The solution to this problem is facilitated by the ***principle of superposition***

→ The interaction between any two charges is **completely unaffected by the others**.

→ To determine the force on Q, we can first compute the force F_1, due to q_1 alone (ignoring all the others); then we compute the force F_2, due to q_2 alone; and so on.

→ Finally, we take the vector sum of all these individual forces: $F = F_1 + F_2 + F_3 + ...$

To solve the force on Q using the superposition principle sounds very easy, BUT,

→ the force on Q depends not only on the separation distance r between the charges,

→ it also depends on *both* their velocities and on the *acceleration* of *q.*

→ ***Moreover,*** electromagnetic fields travels at the speed of light, so the position, velocity, and acceleration that *q had* at some earlier time, affect the force at present.

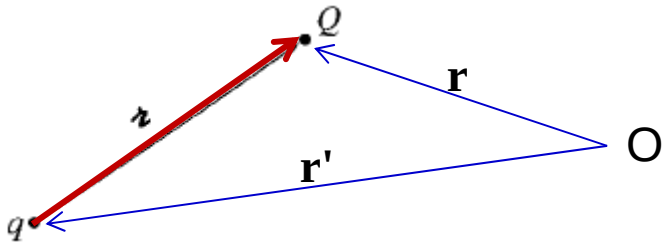
To begin with, **consider the special case of ELECTROSTATICS**

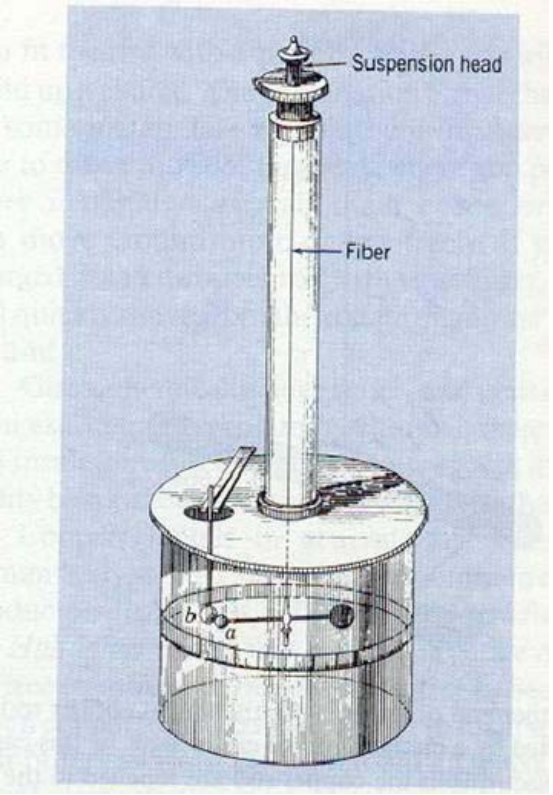
→ ***All the source charges are STATIONARY***

→ ***The test charge may be moving.***

2.1.2 Coulomb's Law

What is the force on a test charge Q due to a single point charge *q* which is at *rest* a distance *r* away?



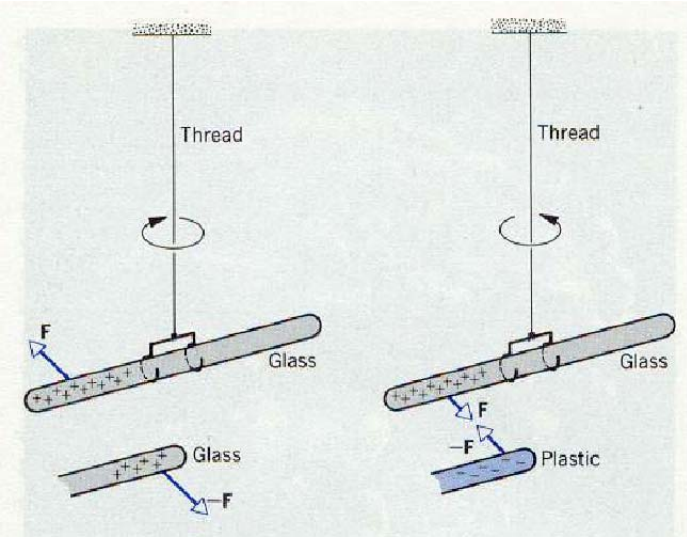


The experimental law of Coulomb (1785)

$$\mathbf{F} = \frac{1}{4\pi\epsilon_0}\frac{qQ}{r^2}\hat{\boldsymbol{r}}$$

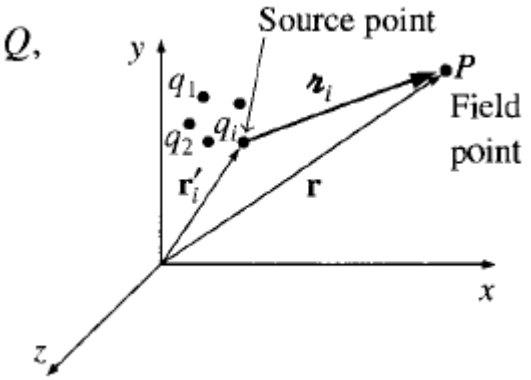
$\boldsymbol{r} = \mathbf{r} - \mathbf{r}'$: separation vector from r' to r

$\epsilon_0 = 8.85 \times 10^{-12}\,\frac{\mathrm{C}^2}{\mathrm{N}\cdot\mathrm{m}^2}$: permitivity of free space



2.1.3 The Electric Field

If we have *several* point charges $q_1, q_2, \ldots, q_n$, at distances $r_1, r_2, \ldots, r_n$ from Q, the total force on Q is evidently



$$\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 + \ldots = \frac{1}{4\pi\epsilon_0}\left(\frac{q_1 Q}{r_1^2}\hat{\mathbf{r}}_1 + \frac{q_2 Q}{r_2^2}\hat{\mathbf{r}}_2 + \ldots\right)$$

$$= \frac{Q}{4\pi\epsilon_0}\left(\frac{q_1\hat{\mathbf{r}}_1}{r_1^2} + \frac{q_2\hat{\mathbf{r}}_2}{r_2^2} + \frac{q_3\hat{\mathbf{r}}_3}{r_3^2} + \ldots\right)$$

$$\mathbf{F} = Q\mathbf{E}$$

$$\mathbf{E}(\mathbf{r}) \equiv \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{n}\frac{q_i}{r_i^2}\hat{\mathbf{r}}_i$$

➔ **E** is called the **Electric Field** of the source charges.

➔ Physically, **E(r) is the force per unit charge** that would be exerted on a test charge, if you were to place one at *P.*

→ Notice that it is **a function of position (r)**, because the separation vectors *r* depend on the location of the field point *P.*

→ It makes no reference to the test charge Q.

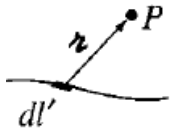
2.1.4 Continuous Charge Distributions

If the charge is distributed continuously over some region, the sun becomes an integral:

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{1}{r^2}\hat{\mathbf{r}}\,dq$$

➔ If the charge is spread out along a *line* with **charge-per-unit-length** λ,

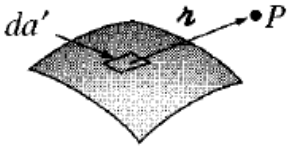
$$dq = \lambda\,dl' \longrightarrow \mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int_{\mathcal{P}} \frac{\lambda(\mathbf{r}')}{r^2}\hat{\mathbf{r}}\,dl'$$



Line charge, λ

➔ If the charge is smeared out over a *surface* with **charge-per-unit-area** σ,

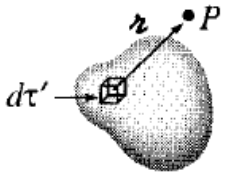
$$dq = \sigma\,da' \longrightarrow \mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int_{\mathcal{S}} \frac{\sigma(\mathbf{r}')}{r^2}\hat{\mathbf{r}}\,da'$$



Surface charge, σ

➔ If the charge fills a *volume* with **charge-per-unit-volume** ρ,

$$dq = \rho\,d\tau' \longrightarrow \mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int_{\mathcal{V}} \frac{\rho(\mathbf{r}')}{r^2}\hat{\mathbf{r}}\,d\tau'$$



Volume charge, ρ

@ **Please note carefully the meaning of *r* in these formulas.**

→ Originally ***r*** stood for the vector from the source charge q_i to the field point **r**.

→ Correspondingly, ***r*** is the vector from *dq* (therefore from *dl', da',* or *dτ')* to the field point **r**.

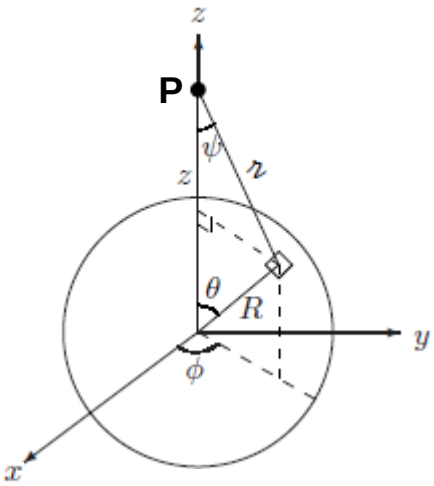
2.1.2 Coulomb's Law

Problem 2.7 Find the electric field a distance z from the center of a spherical surface of radius *R*, which carries a uniform surface charge density σ.

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int_{\mathcal{S}} \frac{\sigma(\mathbf{r}')}{\mathcal{r}^2}\hat{\boldsymbol{\mathcal{r}}}\, da'$$

$$dq = \sigma da = \sigma R^2 \sin\theta\, d\theta\, d\phi$$

$$\mathcal{r}^2 = R^2 + z^2 - 2Rz\cos\theta$$



$$E(\vec{r}) = E_z\hat{z};$$

$$E_z = \left|E(\vec{r})\right|\cos\psi$$

$$\cos\psi = \frac{z - R\cos\theta}{\mathcal{r}}$$

$$E_z = \frac{1}{4\pi\epsilon_0}\int \frac{\sigma R^2 \sin\theta\, d\theta\, d\phi (z - R\cos\theta)}{(R^2 + z^2 - 2Rz\cos\theta)^{3/2}} \qquad \int d\phi = 2\pi$$

$$= \frac{1}{4\pi\epsilon_0}(2\pi R^2\sigma)\int_0^{\pi} \frac{(z - R\cos\theta)\sin\theta}{(R^2 + z^2 - 2Rz\cos\theta)^{3/2}}\, d\theta$$

$$\text{Let } u = \cos\theta;\ du = -\sin\theta\, d\theta;\ \left\{\begin{array}{l}\theta = 0 \Rightarrow u = +1\\ \theta = \pi \Rightarrow u = -1\end{array}\right\}$$

$$= \frac{1}{4\pi\epsilon_0}(2\pi R^2\sigma)\left[\frac{1}{z^2}\frac{zu - R}{\sqrt{R^2 + z^2 - 2Rzu}}\right]_{-1}^{1} = \frac{1}{4\pi\epsilon_0}\frac{2\pi R^2\sigma}{z^2}\left\{\frac{(z - R)}{|z - R|} - \frac{(-z - R)}{|z + R|}\right\}$$

For $z > R$ (outside the sphere), $E_z = \frac{1}{4\pi\epsilon_0}\frac{4\pi R^2\sigma}{z^2} = \frac{1}{4\pi\epsilon_0}\frac{q}{z^2}$, so $\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{z^2}\hat{\mathbf{z}}$.

For $z < R$ (inside), $E_z = 0$, so $\mathbf{E} = \mathbf{0}$

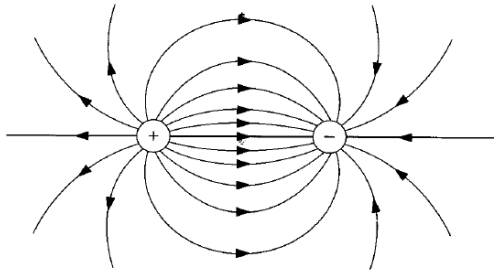
Look! The integrals involved in computing E may be formidable.

➔ **Gauss's Law makes it simple!**

2.2 Divergence and Curl of Electrostatic Fields

2.2.1 Field Lines, Flux, and Gauss's Law

Field lines:

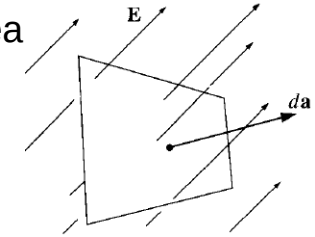

Equal but opposite charges

The magnitude of the field is indicated **by the *density* of the field lines**:

→ it's strong near the center where the field lines are close together,
→ it's weak farther out, where they are relatively far apart.

→ The filed lines begin on positive charges and end on negative ones;
→ they cannot simply terminate in midair, though they may extend out to infinity;
→ they can never cross.

Field flux: $\Phi_E \equiv \int_S \mathbf{E} \cdot d\mathbf{a}$ **→ the "number of field lines" passing through *S*.**



→ **(E** . d**a)** is proportional to the number of lines passing through the infinitesimal area
→ The dot product picks out the component of ***da*** along the direction of **E**,
→ It is only **the area in the plane perpendicular to E.**

→ The number of field lines is proportional to the magnitude of a charge.

→ **Therefore, this suggests that**
"the flux through any *closed* surface is a measure of the total charge inside"
➔ This is the essence of Gauss's Law.

Flux and Gauss's Law

In the case of **a point charge *q*** at the origin, the flux of E through a sphere of radius *r* is

$$\Phi_E \equiv \oint \mathbf{E} \cdot d\mathbf{a} = \int \frac{1}{4\pi\epsilon_0}\left(\frac{q}{r^2}\hat{\mathbf{r}}\right) \cdot (r^2 \sin\theta \, d\theta \, d\phi \, \hat{\mathbf{r}}) = \frac{1}{\epsilon_0} q$$

➔ *The flux through any surface enclosing the charge is q/ε_0.*

Now suppose **a bunch of charges** scattered about.
➔ According to the **principle of superposition**, the total field is the (vector) sum of all the individual fields:

$$\mathbf{E} = \sum_{i=1}^{n} \mathbf{E}_i$$

$$\Phi_E \equiv \oint \mathbf{E} \cdot d\mathbf{a} = \sum_{i=1}^{n} \left(\oint \mathbf{E}_i \cdot d\mathbf{a} \right) = \sum_{i=1}^{n} \left(\frac{1}{\epsilon_0} q_i \right) \Rightarrow \boxed{\oint_{\mathcal{S}} \mathbf{E} \cdot d\mathbf{a} = \frac{1}{\epsilon_0} Q_{\text{enc}}}$$

Gauss's Law

By applying the divergence theorem:

$$\oint_{\mathcal{S}} \mathbf{E} \cdot d\mathbf{a} = \int_{\mathcal{V}} (\nabla \cdot \mathbf{E}) \, d\tau$$

$$Q_{\text{enc}} = \int_{\mathcal{V}} \rho \, d\tau$$

$$\Rightarrow \int_{\mathcal{V}} (\nabla \cdot \mathbf{E}) \, d\tau = \int_{\mathcal{V}} \left(\frac{\rho}{\epsilon_0} \right) d\tau \Rightarrow \boxed{\nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0} \rho}$$

Gauss's law in differential form

2.2.2 The Divergence of E

Let's calculate the divergence of E directly from the Coulomb' s Law of

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int_{\mathcal{V}} \frac{\rho(\mathbf{r}')}{\imath^2}\hat{\boldsymbol{\imath}}\, d\tau' \longrightarrow \nabla\cdot\mathbf{E}$$ (divergence in terms of **r**)

Since the **r**-dependence is contained in $\boldsymbol{\imath} = \mathbf{r} - \mathbf{r}'$, we have

$$\nabla\cdot\mathbf{E} = \frac{1}{4\pi\epsilon_0}\int \nabla\cdot\left(\frac{\hat{\boldsymbol{\imath}}}{\imath^2}\right)\rho(\mathbf{r}')\, d\tau'$$

$$\nabla\cdot\left(\frac{\hat{\boldsymbol{\imath}}}{\imath^2}\right) = 4\pi\delta^3(\boldsymbol{\imath})$$

$$\nabla\cdot\mathbf{E} = \frac{1}{4\pi\epsilon_0}\int 4\pi\delta^3(\mathbf{r}-\mathbf{r}')\rho(\mathbf{r}')\, d\tau' = \frac{1}{\epsilon_0}\rho(\mathbf{r})$$

$$\nabla\cdot\mathbf{E} = \frac{\rho(r)}{\varepsilon_0}$$

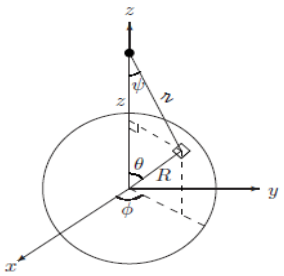
➔ ***This is Gauss's law in differential form***

2.2.3 Applications of Gauss's Law $\oint_{\mathcal{S}} \mathbf{E} \cdot d\mathbf{a} = \frac{1}{\epsilon_0} Q_{\text{enc}}$

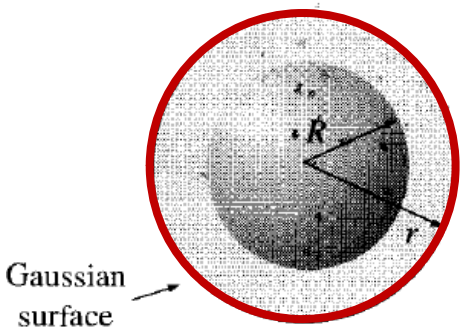
When symmetry permits, it's the quickest and easiest way of computing electric fields.

Example 2.2 Find the field outside a uniformly charged solid sphere of radius *R* and total charge *q*. (Remember the Prob. 2.7)

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{z^2}\hat{\mathbf{z}}.$$



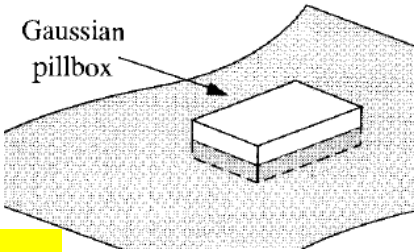
➔ Draw a spherical surface at radius $r > R$ ➔ "**Gaussian surface**"



$$\oint_{\mathcal{S}} \mathbf{E} \cdot d\mathbf{a} = \frac{1}{\epsilon_0} Q_{\text{enc}}$$

$$\int_{\mathcal{S}} |\mathbf{E}|\, da = |\mathbf{E}| \int_{\mathcal{S}} da = |\mathbf{E}|\, 4\pi r^2 = \frac{1}{\epsilon_0} q \quad \Rightarrow \quad \mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{\mathbf{r}}$$

the magnitude of E is constant over the Gaussian surface



Gauss's law is always *true,* ***but it is not always useful.***

➔ ***Symmetry is crucial*** to this application of Gauss's law.

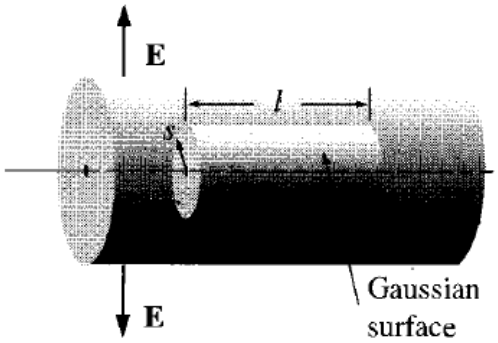
1. Spherical symmetry. **Make your Gaussian surface a concentric sphere.**
2. Cylindrical symmetry. **Make your Gaussian surface a coaxial cylinder.**
3. Plane symmetry. **Use a Gaussian "pillbox," which straddles the surface.**

Applications of Gauss's Law $\oint_S \mathbf{E} \cdot d\mathbf{a} = \frac{1}{\epsilon_0} Q_{\text{enc}}$

Example 2.3 A long cylinder carries a charge density that is proportional to the distance from the axis: $\rho = ks$, for some constant k.

➔ The enclosed charge is

$$Q_{\text{enc}} = \int \rho \, d\tau = \int (ks')(s' \, ds' \, d\phi \, dz) = 2\pi kl \int_0^s s'^2 \, ds' = \tfrac{2}{3}\pi kls^3$$



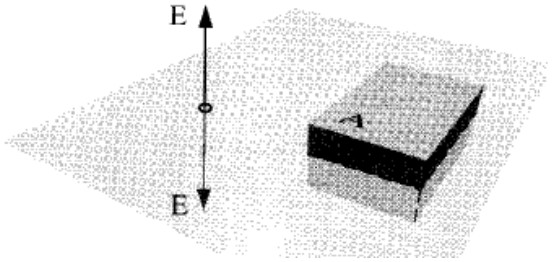
Symmetry dictates that **E** must point radially outward,

$$\int \mathbf{E} \cdot d\mathbf{a} = \int |\mathbf{E}| \, da = |\mathbf{E}| \int da = |\mathbf{E}| \, 2\pi sl$$

While the two ends contribute nothing (here **E** is perpendicular to $d\mathbf{a}$). Thus,

$$|\mathbf{E}| \, 2\pi sl = \frac{1}{\epsilon_0} \frac{2}{3} \pi kls^3 \longrightarrow \boxed{\mathbf{E} = \frac{1}{3\epsilon_0} ks^2 \hat{\mathbf{s}}}$$

Example 2.4 An infinite plane carries a **uniform surface charge** σ.



➔ Draw a **"Gaussian pillbox,"** and apply Gauss's law to this surface:

$$Q_{\text{enc}} = \sigma A$$

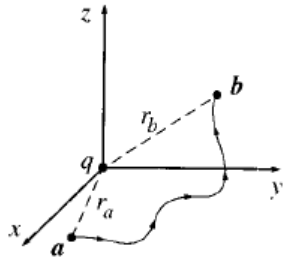
By symmetry, E points away from the plane (upward for points above, downward *for* points below)

$$\int \mathbf{E} \cdot d\mathbf{a} = 2A|\mathbf{E}| = \frac{1}{\epsilon_0} \sigma A \longrightarrow \boxed{\mathbf{E} = \frac{\sigma}{2\epsilon_0} \hat{\mathbf{n}}}$$

2.2.4 The Curl of E

Consider the electric field from a point charge **q** at the origin: $\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{\mathbf{r}}$

Now let's calculate the line Integral of this field from some point **a** to some other point **b:** $\int_{\mathbf{a}}^{\mathbf{b}} \mathbf{E}\cdot d\mathbf{l}$



In spherical coordinates, $d\mathbf{l} = dr\,\hat{\mathbf{r}} + r\,d\theta\,\hat{\boldsymbol{\theta}} + r\sin\theta\,d\phi\,\hat{\boldsymbol{\phi}}$

$$\mathbf{E}\cdot d\mathbf{l} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}dr \longrightarrow \int_{\mathbf{a}}^{\mathbf{b}} \mathbf{E}\cdot d\mathbf{l} = \frac{1}{4\pi\epsilon_0}\int_{\mathbf{a}}^{\mathbf{b}} \frac{q}{r^2}dr = \frac{-1}{4\pi\epsilon_0}\frac{q}{r}\Big|_{r_a}^{r_b} = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{r_a} - \frac{q}{r_b}\right)$$

The integral around a *closed* path is evidently zero (for then $r_a = r_b$*):*

$\oint \mathbf{E}\cdot d\mathbf{l} = 0$ ➔ Applying Stokes' theorem, $\nabla\times\mathbf{E} = 0$

If we have many charges, the principle of superposition states that the total field is a vector sum of their individual fields: $\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2 + \ldots,$

$$\nabla\times\mathbf{E} = \nabla\times(\mathbf{E}_1 + \mathbf{E}_2 + \ldots) = (\nabla\times\mathbf{E}_1) + (\nabla\times\mathbf{E}_2) + \ldots = 0$$

$\nabla\times\mathbf{E} = 0$ ➔ *For any static charge distribution whatever*